

ADDENDUM TO THE PAPER "EXISTENCE OF WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS WITH INITIAL DATA IN L^p "

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ABSTRACT. This paper considers the existence of global weak solutions for the Navier-Stokes equations in the infinite cylinder $\mathbf{R}^n \times \mathbf{R}_+$ with initial data in L^r , $n \geq 3$, $1 < r < \infty$. An imbedding theorem as well as related initial value problems are also studied, thus completing results in [2].

INTRODUCTION

This paper considers the initial value problem for the Navier-Stokes equations in the infinite cylinder $S_T = \mathbf{R}^n \times [0, T)$. Given $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, satisfying in the distributions sense $\operatorname{div} f = 0$, $x \in \mathbf{R}^n$, we seek a solution vector $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ and a pressure function $P(x, t)$ such that

$$(0.1) \quad \begin{aligned} D_t u_i - \sum_{j=1}^n D_{jj} u_i + \sum_{j=1}^n u_j D_j u_i + D_i P &= 0, & (x, t) \in S_T, \\ \sum_j D_j u_j &= 0, & (x, t) \in S_T, \\ u(x, 0) &= f(x). \end{aligned}$$

Here, D_j and D_t denote respectively, the distributional derivatives with respect to x_j and t , D_{ij} denotes the second order derivative with respect to x_i, x_j ; likewise, $L(u)$ will denote the heat operator applied to u , and $\operatorname{grad} u$, the square matrix $D_j u_i$. The first equation of (0.1) takes the form

$$(0.2) \quad L(u) + (\operatorname{grad} u)(u) + \operatorname{grad} P = 0.$$

Following [4], I consider the functional spaces $L^{p,q}(S_T)$ consisting of the Lebesgue-measurable functions u , such that

$$(0.3) \quad \|u\|_{p,q}(T) < \infty.$$

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Let us define $U_p(t)$ as

$$(0.4) \quad \left| \int_{R^n} |u|^p dx \right|^{1/p},$$

Then, the mixed norm $(0, 3)$ can be written as

$$(0.5) \quad \|U_p\|_q(T) \quad (\text{usual } L^q\text{-norm over the interval } (0, T)).$$

The norms associated with the maximal operator $u^* = \sup_t |u|$ (the supremum is taken over $t > 0$) are

$$(0.6) \quad \|u^*\|_p(T) = \left(\int \left(\sup_{0 < t < T} |u| \right)^p dx \right)^{1/p}.$$

In the particular case when we take $T = \infty$, we have

$$(0.7) \quad \|u^*\|_p(\infty) = \left(\int \left(\sup_{t > 0} |u| \right)^p dx \right)^{1/p}.$$

In the same fashion we introduce the norms $\| \cdot \|_{p,q}^*$ as

$$(0.8) \quad \|u\|_{p,q}^* = \left(\int_{R^n} \left(\int_0^\infty |u(x, t)|^p dt \right)^{q/p} dx \right)^{1/q}.$$

The aim of this paper is to complete the results in [2] concerning solutions of $(0, 1)$ for initial data in $L^n(\mathbf{R}^n)$. Likewise, in §IV below, a relation is established between the $L^{p,q}$ classes of existence and uniqueness introduced by Fabes, Jones, Riviere in [4] and the classes of solutions with initial data in $L^r(\mathbf{R})$.

I. MAIN RESULTS

Weak solutions [4]. A function $u(x, t)$ is said to be a weak solution of the Navier-Stokes equations, with initial values f , $\operatorname{div} f = 0$ (in the distributions sense), if for any C^∞ , rapidly decreasing vector function $v(x, t) = (v_1(x, t), \dots, v_n(x, t))$, defined on $R^{n+1}(x, t)$, such that $\operatorname{div} v = 0$, $v(x, t) = 0$, $t > T$, we have

(a) $u \in L^{p,q}(S_T)$ with $p, q \geq 2$,

(b) $\int_0^T \int_{R^n} \langle u, L^*(v) + (\operatorname{grad} v)(u) \rangle dx dt = - \int_{R^n} \langle f(x), v(x, 0) \rangle dx$ where L^* is the adjoint heat operator,

(c) $\operatorname{div} u(x, t) = 0$ (in the distributions sense) for a.e. t , such that $0 < t < T$.

Theorem A. Let n be greater than or equal to 3 and $f(x)$, the initial data, a vector function, such that $f(x) \in L^r(\mathbf{R}^n)$, $1 < r < \infty$, and $\operatorname{div} f = 0$ in the distributions sense. Let $F(x, t) = W * f$, where W is the fundamental solution of the heat equation (see II below); the convolution is taken in the spatial coordinates. Then

$$(i) \quad \|F\|_{p,q}^{1/2} < C_{p,q} \|f\|_r, \quad \frac{n}{r} = \frac{n}{p} + \frac{2}{q}, \quad q \geq p,$$

¹ $\|F\|_{p,q} = \|F\|_{p,q}(S_\infty)$.

- (ii) $\|F\|_{p,q}^* < C_{p,q} \|f\|_r$, $\frac{n}{r} = \frac{2}{p} + \frac{n}{q}$, $p \geq q$,
 (iii) If for some p, q , $1 = \frac{n}{p} + \frac{2}{q}$, $n < p$, we have
- $$\|F\|_{p,q} < \varepsilon_{p,q}(n),$$

where $\varepsilon_{p,q}(n)$ is a fixed small quantity depending on p, q and n only. Then, there exists a unique solution u to the problem $(0, 1)$, that is global and satisfies the equations $(0, 1)$ and the initial data in the weak sense. The uniqueness holds in the class of functions u such that

$$\|u\|_{p,q} < \infty.$$

Likewise, if for some (p, q) ; $1 = \frac{n}{q} + \frac{2}{p}$; $n < q$, we have

$$\|F\|_{p,q}^* < \varepsilon_{p,q}(n)$$

where $\varepsilon_{p,q}(n)$ is a small quantity depending on p, q and n only; then, there exists a weak solution u of $(0, 1)$ for the initial data f , that satisfies $(0, 1)$ for all time $t > 0$. The solution u is unique in the class of functions that satisfy $\|u\|_{p,q}^* < \infty$.

Corollary. Let the initial data f belong to $L^n(\mathbf{R}^n)$. If for some $L^{p,q}$ -norm we have

$$\|F\|_{p,q} < \varepsilon_{p,q}(n), \quad 1 = \frac{n}{p} + \frac{2}{q}, \quad p < q.$$

Then, there exists a weak, global solution for the problem $(0, 1)$. The solution is unique in the corresponding class of functions $L^{p,q}(S_\infty)$.

II. FUNDAMENTAL SOLUTIONS, THE BILINEAR FORM, MIXED POINTS

Fabes-Jones-Riviere, [4], extended to dimension n a formula found for the case $n = 3$ by Oseen [10]. The Oseen-Fabes-Jones-Riviere formula gives a divergence free matrix fundamental solution $E_{ij}(x, t)$ for an n -dimensional heat equation. The matrix $E_{ij}(x, t)$ is defined in the following way:

$$(2.1) \quad E(x, t) = \delta_{ij} W(x, t) - R_i R_j W(x, t),$$

where $W(x, t) = (4\pi t)^{-n/2} \exp -|x|^2/(4t)$, and R_j is the j th Riesz transform, namely,

$$(2.2) \quad R_j(f) = \text{p.v. } c_j \int (x_j - y_j) |x - y|^{-(n+1)} f(y) dy.$$

For details, see [4 and 13]. $E_{ij}(x, t)$ is symmetric and divergence free, that is,

$$(2.3) \quad \sum_j D_j E_{ij}(x, t) = 0, \quad t \geq 0.$$

In the above formula we take classical derivatives if $t > 0$, and distributional derivatives for the limit for t tending to 0. An other important property is the following one:

$$\sum_j \int E_{ij}(x, t) f_j(y - x) dx \text{ tends to } f_i(y) \text{ in } L^p, \quad 1 < p < \infty, \\ \text{as } t \text{ tends to zero, provided that } \operatorname{div} f = 0.$$

A very important theorem in [4] asserts that $u(x, t)$ is a weak solution of the problem $(0, 1)$ (see definition in §I) over S_T , with $g \in L^r$, $1 \leq r < \infty$, if and only if it is a solution of the following integral equation [4, Theorem 2.1, p. 226]:

$$(2.4) \quad u + B(u, u) = F(x, t).$$

$F(x, t)$ stands for the convolution in the space variables of the initial data $f(x)$ with the fundamental solution $W(x, t)$. $B(u, v)$ is the bilinear form

$$(2.5) \quad \int_0^t \int_{R^n} \langle (\text{grad } E(x - y, t - s))(v(y, s)), u(y, s) \rangle dy ds.$$

We have used here a notation consistent with $(0, 2)$. In fact, $(\text{grad } E)(v)$ is the matrix $\sum_k D_k E_{ij} v_k$, whose i th row is dotted with u to obtain the integrand of (2.5).

Estimates for the bilinear operators and fixed-point properties.

In what follows, we are going to consider Banach spaces of Lebesgue measurable functions defined on S_T for which the operator $T(u, v) = B(u, v) + 1(u) + F(x, t)$, ($B(u, v)$ is bilinear and $1(u)$ is assumed to be linear) satisfies an inequality of the type

$$(2.6) \quad \|T(u, v)\| \leq C_1 \|u\| \|v\| + C_2 \|u\| + \|F\|.$$

In (2.9) above, the norm is that of the Banach space in question.

Lemma II. *The quadratic operator $T(u, u)$ maps the ball $\{\|u\| \leq s_1\}$ into itself if s_1 is the smallest root of the equation*

$$(2.7) \quad C_1 s^2 + (C_2 - 1)s + \|F\| = 0$$

provided that C_1, C_2 and $\|F\|$ satisfy

$$(2.8) \quad (1 - C_2)^2 > 4C_1 \|F\|, \quad C_1 > 0, \quad 0 \leq C_2 < 1.$$

If $2s_1 C_1 + C_2 < 1$, $T(u, u)$ is a contraction mapping in the ball of radius s_1 . In particular, $T(u, u)$ is a contraction mapping in the ball of radius s_1 if C_1, C_2 and $\|F\|$ satisfy

$$(2.9) \quad 2C_1 \|F\| \{(1 - C_2)^2 - 4C_1 \|F\|\}^{-1/2} + C_2 < 1.$$

For the proof see [2].

III. ESTIMATES FOR THE BILINEAR FORM $B(u, u)$

The bilinear form $B(u, v)$ admits the domination

$$(3.1) \quad |B(u, v)| \leq C \int_{R^n} |x - y|^{-n+1} \int_0^t |x - y|^{-2} (1 + |x - y|^{-1} s^{1/2})^{-n-1} \\ \times |u(y, t - s)| |v(y, t - s)| ds dy.$$

The above domination is a consequence of the estimate

$$(3.2) \quad |D_k E_{ij}(x, t)| \leq C(|x| + t^{1/2})^{-n-1}.$$

Calling $M(u)$ the maximal function of Hardy-Littlewood of $|u|$ on the space variables and u^* the sup on $t > 0$ of $|u|$, we have

$$(3.3) \quad |B(u, u)| \leq C_0 \int_{\mathbb{R}^n} |x - y|^{-n+1} (M(u^*))^2 dy.$$

The constant C_0 does not exceed

$$(3.4) \quad C \int_0^\infty (1 + t^{1/2})^{-n-1} dt.$$

The right-hand side of (3.14) does not depend on t , hence

$$(3.5) \quad |B(u, u)|^* \leq C_0 \int_{\mathbb{R}^n} |x - y|^{-n+1} (M(u^*))^2 dy.$$

We now apply Hardy-Littlewood-Sobolev potential inequality to (3.5) with exponents $\frac{1}{q} = \frac{2}{n} - \frac{1}{n}$ (see [13, pp. 119, 120]) and obtain

$$(3.6) \quad \|B(u, u)^*\|_n(\infty) \leq C \{ \|M(u^*)\|_n(\infty) \}^2 \leq C' \{ \|u^*\|_n(\infty) \}^2.$$

We now need estimates on $D_k E_{ij}$ of the type introduced by Benedek-Panzone in [1] (see p. 321, Theorem 1), namely,

$$(3.5) \quad |D_k E_{ij}| \leq \frac{C}{|x|^{n-\theta} (t^{1/2})^{1+\theta}}.$$

Here, C is an independent constant, $0 < \theta < 1$, $t > 0$, $x \in \mathbb{R}^n$. A simple adaptation of Theorem 1, p. 321 in [1] (see also [4, Theorem (3.1)]) gives for the operator

$$T(f) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\theta} (t - \tau)^{\frac{1+\theta}{2}}} f(y, \tau) dy d\tau$$

the estimates

$$(3.7) \quad \|T(f)\|_{p^*, q^*} \leq C_{p, q} \|f\|_{p, q}$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{\theta}{n} \frac{1}{q^*} = \frac{1}{q} - \frac{1-\theta}{2}.$$

Similar results hold for the norms $\| \cdot \|_{p, q}^*$, namely,

$$(3.8) \quad \|T(f)\|_{p^*, q^*}^* \leq C_{p, q} \|f\|_{p, q}^*.$$

Here,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1-\theta}{2} \frac{1}{q^*} = \frac{1}{q} - \frac{\theta}{n}.$$

The estimates (3.5), (3.7), (3.8) and (3.6) lead to the following inequalities for $B(u, u)$:

$$(3.9) \quad \|B(u, u)\|_{\frac{n}{\theta}, \frac{2}{1-\theta}} \leq C_\theta (\|u\|_{\frac{n}{\theta}, \frac{2}{1-\theta}})^2,$$

$0 < \theta < 1$, which is a consequence of

$$(3.10) \quad |B(u, u)| \leq CT(|u|^2).$$

T as defined in (3.6) above. Likewise we get for the $\|\cdot\|_{p,q}^*$ norms the estimate

$$(3.11) \quad \|B(u, u)\|_{\frac{2}{1-\theta}, \frac{n}{\theta}}^* \leq C_\theta (\|u\|_{\frac{2}{1-\theta}, \frac{n}{\theta}}^*)^2,$$

$0 < \theta < 1$. In the above expressions we may replace $\frac{n}{\theta}$ and $\frac{2}{1-\theta}$ by p and q respectively, satisfying

$$1 = \frac{n}{p} + \frac{2}{q}; \quad p > n.$$

IV. IMBEDDING THE INITIAL DATA

As we have seen in (2.4), $F(x, t)$ is the term arising from the initial data:

$$(4.1) \quad F(x, t) = W * f.$$

The above convolution is on the spatial variables only. The "a priori estimate" (3.86) [2] and Lemma A₃ in [2] give the following result as a trivial consequence:

Lemma IV. *Let $1 \leq n < \infty$, $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. Then, the function $F(x, t)$ as defined above, satisfies*

$$(i) \quad \|F\|_{\frac{n+2}{n}p, \frac{n+2}{n}p} \leq C_p(n) \|f\|_p,$$

$$(ii) \quad \|F\|_{\frac{n+2}{n}p, \frac{n+2}{n}p}^* \leq C_p(n) \|f\|_p.$$

$C_p(n)$ depends only on p and n .

Theorem B. *The function $F(x, t)$ defined above satisfies*

$$(i) \quad \|F\|_{s,v} \leq C_r \|f\|_r,$$

$$(ii) \quad \|F\|_{v,s}^* \leq C_r \|f\|_r, \quad 1 < r < \infty, \quad v > s, \quad \frac{1}{r} = \frac{1}{s} + \frac{2}{nv}.$$

C_r depends only on r, v, s and n .

Proof. We shall consider the norms $\|\cdot\|_{p,q}$ only, since $\|\cdot\|_{p,q}^*$ can be dealt with in a similar manner ($\tilde{u}(t, x) = u(x, t)$, thus $\|\tilde{u}\|_{p,q} = \|u\|_{p,q}^*$). On one hand we have

$$\|F\|_{p,\infty} < C_p \|f\|_p, \quad 1 < p \leq \infty,$$

and from Lemma IV

$$\|F\|_{\frac{n+2}{n}q, \frac{n+2}{n}q} < C_q \|f\|_q, \quad 1 < q \leq \infty.$$

The Benedek-Panzone interpolation theorem for mixed norms (see [1, Theorems 1, 2]); gives the desired result for

$$(4.2) \quad \begin{aligned} \frac{1}{r} &= \frac{t}{p} + \frac{1-t}{q}, \\ \frac{1}{s} &= \frac{t}{p} + \frac{1-t}{q} \frac{n}{n+2}, \\ \frac{1}{v} &= \frac{1-t}{q} \frac{n}{n+2}, \quad 0 < t < 1, \end{aligned}$$

or

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{v} + \frac{n+2}{n} \frac{1}{v}, \quad v > s,$$

hence,

$$(4.3) \quad \frac{1}{r} = \frac{1}{s} + \frac{2}{nv}.$$

Concerning the norms $\| \cdot \|_{p,q}^*$, one should notice that

$$\|F\|_{\infty,p}^* < C_p \|f\|_p, \quad 1 < p \leq \infty,$$

is a consequence of the maximal theorem associated with the Weierstrass kernel. This finishes the proof.

V. PROOF OF THEOREM A

Parts (i) and (ii) follow from the imbedding results of §IV. Part (iii) follows using the estimates (3.9) and (3.11) and Lemma II. This concludes the proof.

REFERENCES

1. A. Benedek and R. Panzone, *The spaces L^p with mixed norm*, Duke Math. J. **28** (1961), 301–324.
2. Calixto P. Calderón, *Existence of weak solutions for the Navier-Stokes equations with initial data in L^p* , Trans. Amer. Math. Soc. (to appear).
3. E. B. Fabes and N. M. Riviere, *Singular integrals with mixed homogeneity*, Studia Math. **27** (1966), 19–38.
4. E. B. Fabes, B. F. Jones, and N. M. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p* , Arch. Rational Mech. Anal. **45** (1972), 222–240.
5. E. B. Fabes, J. E. Lewis, and N. M. Riviere, *Singular integrals and hydrodynamical potentials*, Amer. J. Math. **99** (1977), 601–625.
6. ———, *Boundary value problems for the Navier-Stokes equations*, Amer. J. Math. **99** (1977), 626–668.
7. E. Hopf, *Ueber die Anfangsaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951).
8. O. A. Ladyzhenskaja, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.
9. J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
10. C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademie Verlagsgesellschaft, Leipzig, 1927, p. 68.
11. G. Prodi, *Un teorema di unicita per le equazioni di Navier Stokes*, Ann. Mat. **48** (1959), 173–182.
12. J. Serrin, *The initial value problem for the Navier Stokes equations*, Non Linear Problems, edited by R. E. Langer, Univ. of Wisconsin Press, 1963, pp. 69–83.
13. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series, No. 30, Princeton Univ. Press, 1970.
14. F. B. Weissler, *Initial value problem in L^p for the Navier Stokes fluids*, Arch. Rational Mech. Anal. **74** (1980), 219–229.